Chapter Six

Split Graphs

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Introduction

Undirected Graphs

Some Possible Properties:

- Comparability
- Cocomparability
- Triangulated
- Cotriangulated

✓ Independent Properties (Appendix F)

Graph Classes

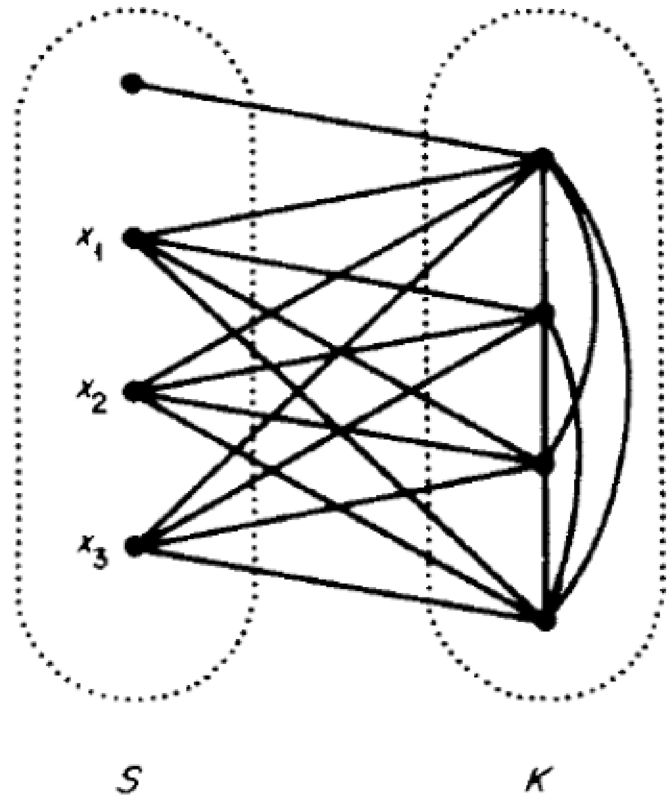
- Interval Graphs: T+C-bar
- Permutation Graphs: C+C-bar
- Split Graphs: T+T-bar

Characteristics

- V = S + K

☒Unique

☒Necessarily Maximal



Theorem 6.1

- G is split if & only if G-bar is split

Theorem 6.2

- Exactly One of the Following Conditions Hold:

➢ $|S| = \alpha(G)$, $|K| = \omega(G)$

The Partition is Unique

➢ $|S| = \alpha(G)$, $|K| = \omega(G)-1$

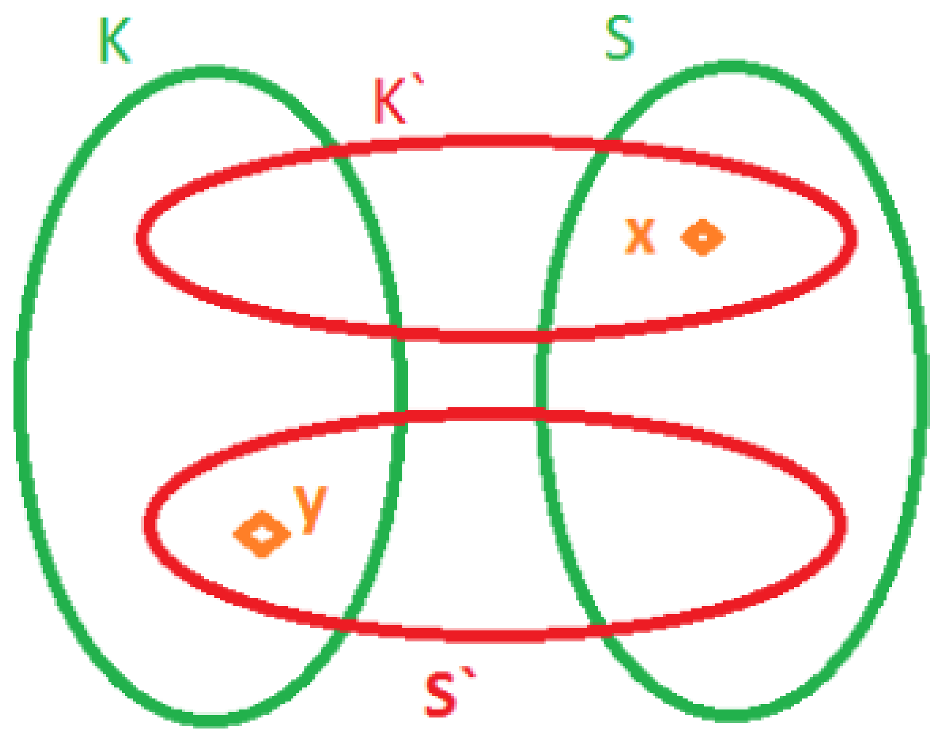
$\exists x \in S: K + \{x\} = \omega(G)$

➢ $|S| = \alpha(G)-1$, $|K| = \omega(G)$

$\exists y \in K: S + \{y\} = \alpha(G)$

Proof of 6.2

- S , K at most one common vertex
- $\alpha(G) + \omega(G) = |V|$ or $|V|+1$



Theorem 6.3

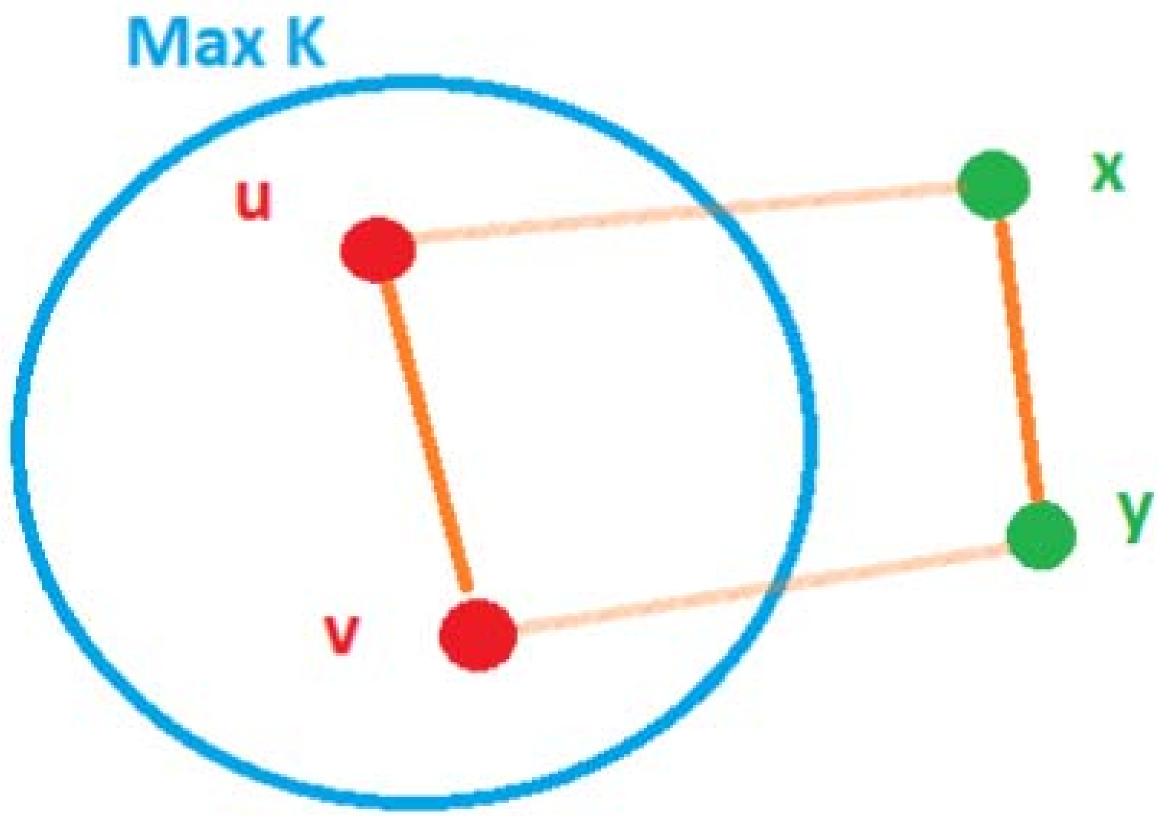
- Following Conditions Are Equivalent:

- G is a split graph

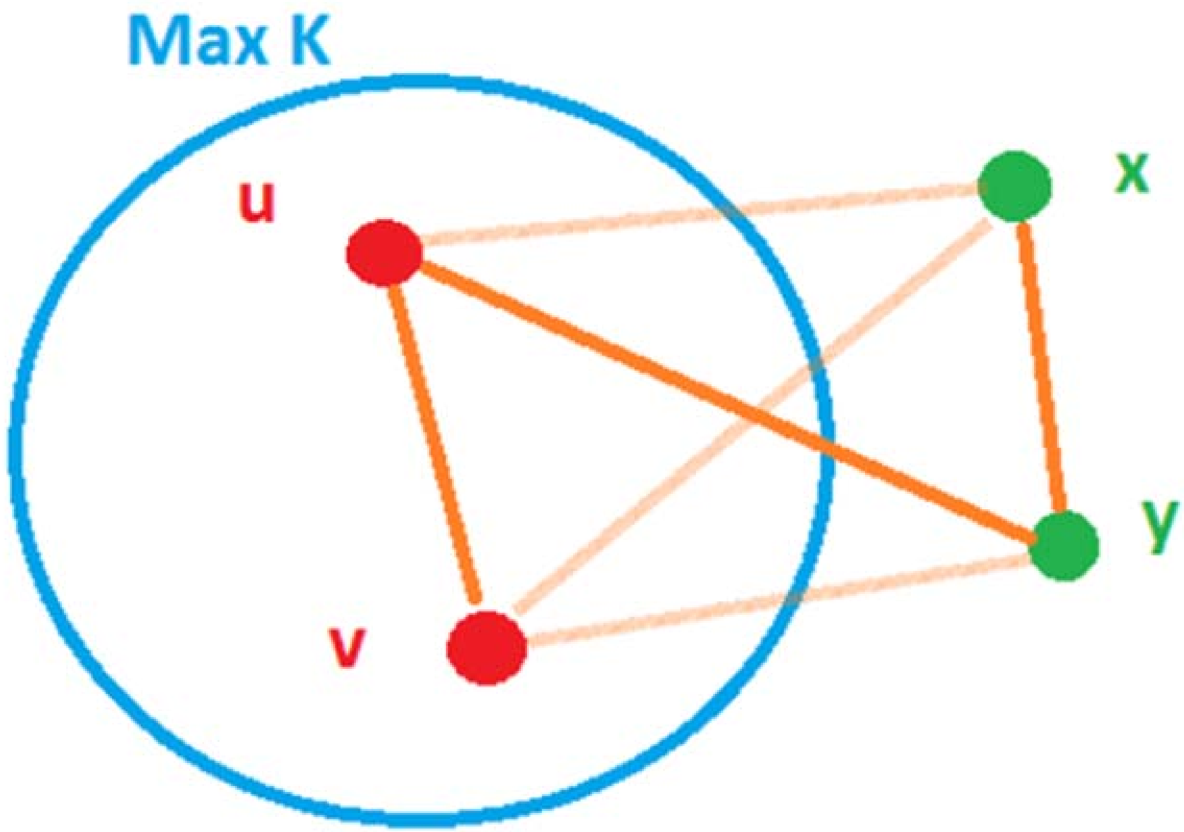
- G and G-bar are triangulated

- G contains no induced subgraph isomorphic to $2K_2$, C_4 , C_5

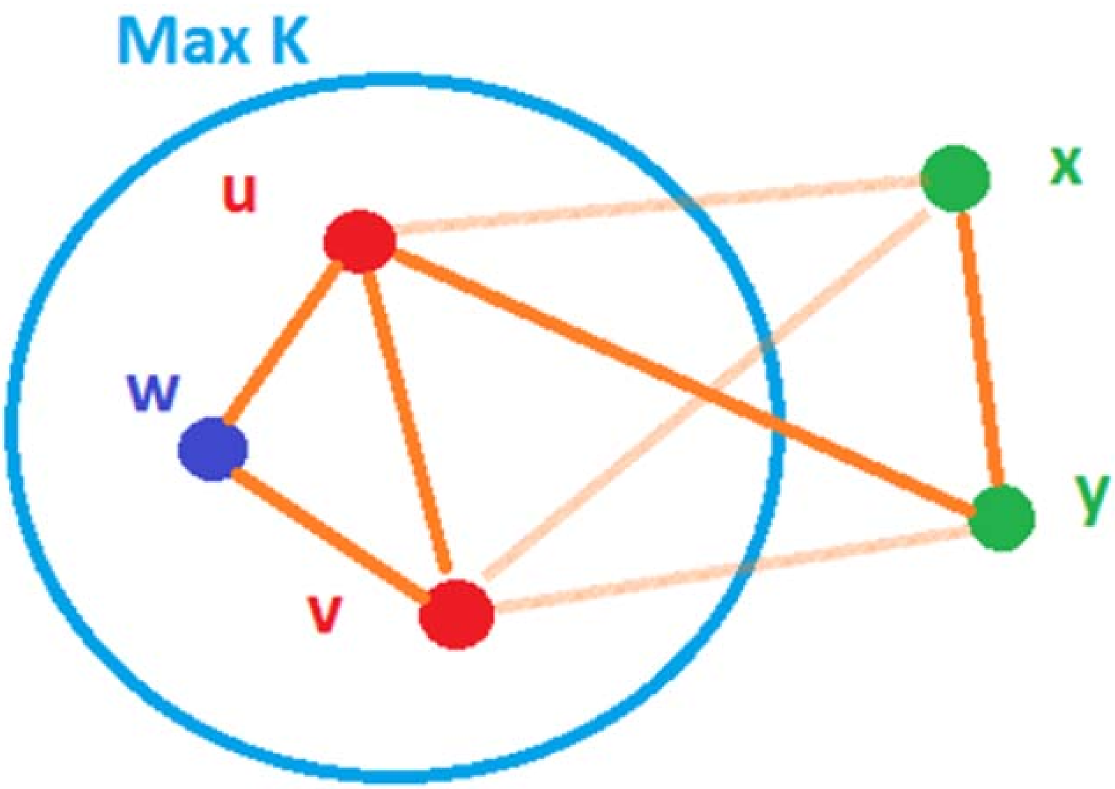
Proof of 6.3



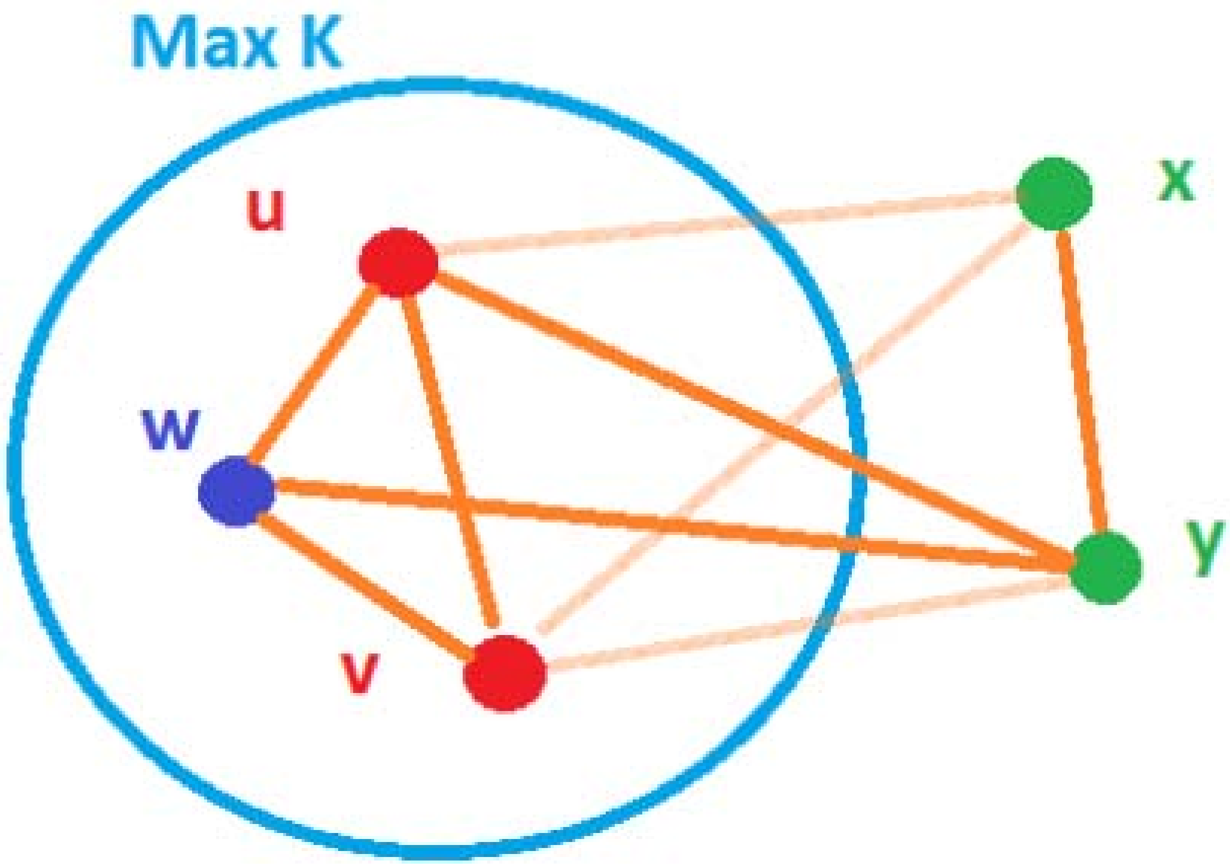
Proof of 6.3



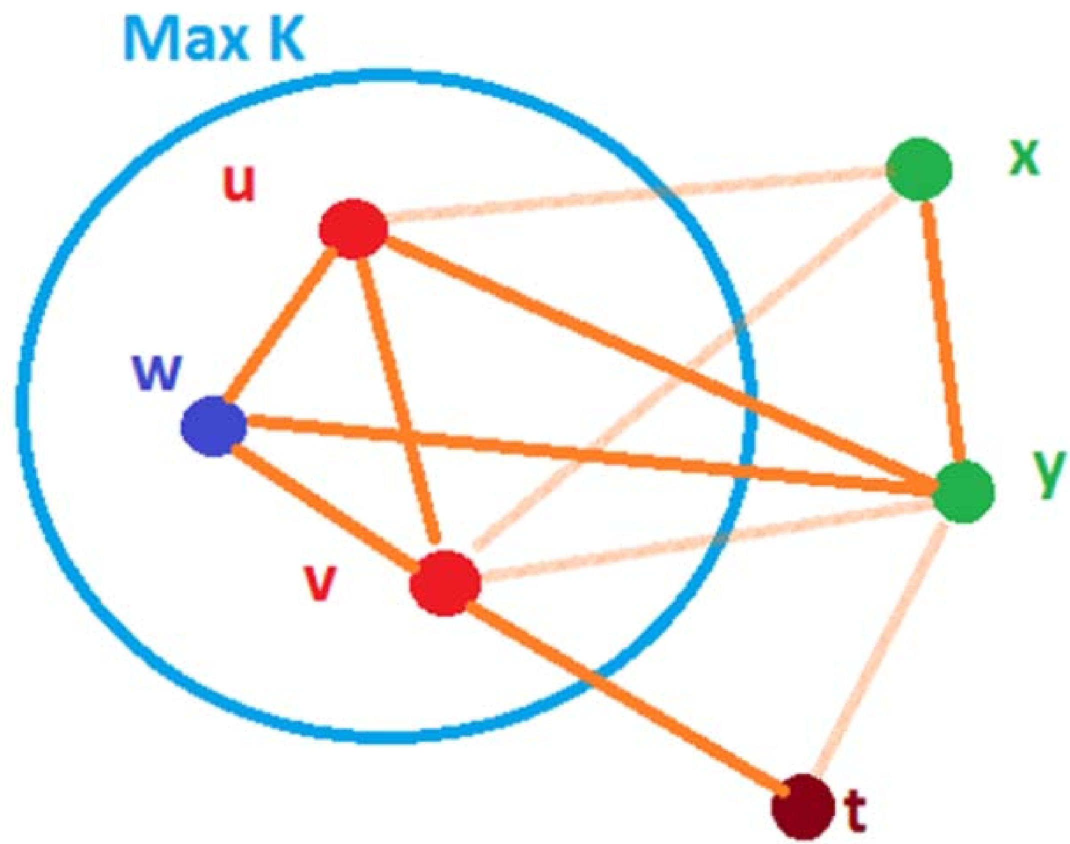
Proof of 6.3



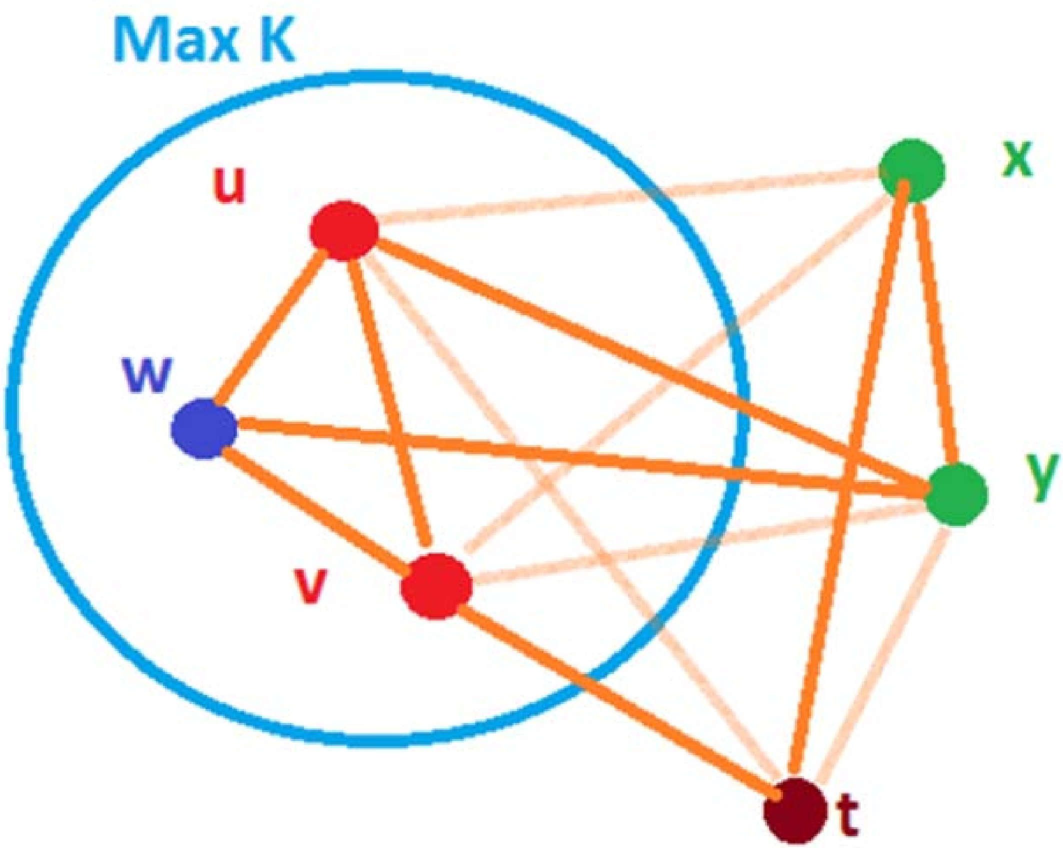
Proof of 6.3



Proof of 6.3



t,x,y,u,v…



Degree Sequence

- $\Delta = [d_1, d_2, \ldots d_n]$

- Euler's Theorem: Even Sum

- Necessary But Insufficient

Theorem 6.4 & 6.5 (NO PROOF)

Degree sequence is graphic if & only if:

1.

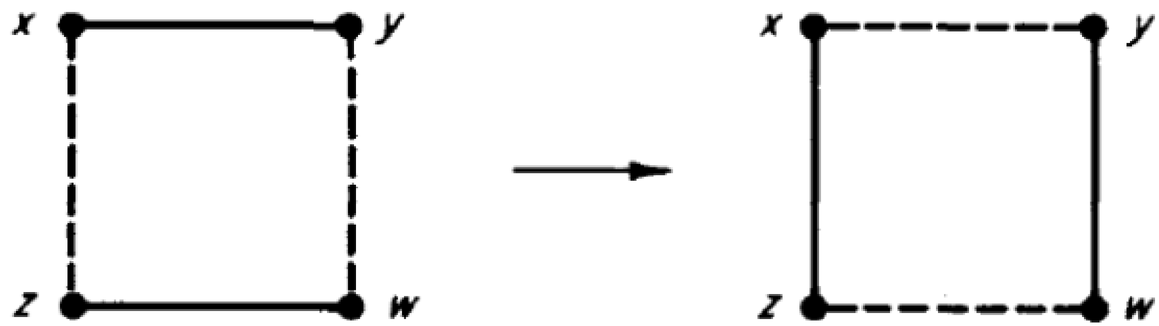
- $n - 1 \geq di \geq 0$ i:{1-n}
- $\Delta` = [d_2 - 1, d_3 - 1, \ldots, d_{d1+1} - 1, d_{d1+2}, \ldots, d_n]$

  is graphic

Or 2.

- r[th] Erdos-Gallai inequality (EGI)
- $\sum_{i=1}^{r} di \leq r(r-1) + \sum_{i=r+1}^{n} \min\{r, di\}$
- r = 1, 2 , … , n-1

Remark 6.6



- Interchange
- **A Strong Result:**
- If two graphs have the same degree sequence then they can be obtained from each other through finite sequence of interchanges.

Theorem 6.7

- $\Delta = [d_1, d_2, \ldots d_n]$
- $\zeta = [0,1,2,\ldots,n\text{-}1]$
- m = overtaking point

- G is a Split Graph if & only if:
- $\sum_{i=1}^{m} di = m(m-1) + \sum_{i=m+1}^{n} di$
- If this is the case then: $\omega(G)=m$

Proof of 6.7

- Theorem is true for complete graphs
- $dm \geq m - 1\,, d(m+1) < m$
- $i \geq m + 1 \;\rightarrow \min\{\mathrm{m,di}\} = \mathrm{di}$
- $s = \sum_{i=1}^{m} di \leq m(m-1) + \sum_{i=m+1}^{n} di$
- $\mathrm{s} = \mathrm{s}_1 + \mathrm{s}_2$

Proof of 6.7

- $s1 = \sum_{x \in K} |\{z \in K | xz \in E\}| \leq m(m-1)$
- $s2 = \sum_{x \in K} |\{y \notin K | xy \in E\}|$

  $= \sum_{x \notin K} |\{x \in K | xy \in E\}| \leq \sum_{i=m+1}^{n} di$
- s_1 holds incase of K's completeness
- s_2 holds only if V-K is stable

- Converse: Size of the max cliques is selected and used for the beginning of the sequence

Corollary 6.8

- If G is a Split Graph; Every Graph with the same degree sequence is also a Split Graph
- Splittance: Min # of edges to be modified on a graph to produce a split graph.
- $\frac{1}{2}[m(m-1) - \sum_{i \leq m} di + \sum_{i \geq m+1} di]$